MTH 166

Lecture-6

Revision of Unit-1 and MCQ Practice

Exact Differential Equation (EDE):

An equation of the form:

$$M(x, y)dx + N(x, y)dy = 0 \qquad (1)$$

is called an exact differential equation if and only if:

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ (Necessary and Sufficient Condition for EDE)

The solution of equation (1) is given by:

$\int_{y=con.} M(x, y)dx + \int (Terms\ of\ N\ free\ from\ x)dy = C$(Constant)

Exact Differential Equation (EDE):

EDE Form:

$$Mdx + Ndy = 0$$

Necessary and Sufficient Condition:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Solution of EDE:

$$\int_{y=con.} Mdx + \int (Terms\ of\ N\ free\ from\ x)dy = C$$

If $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ **,**then Equation $Mdx + Ndy = 0$ is Non-Exact differential equation.

Integrating Factor (I.F.):

An Integrating Factor (I.F.) is a factor which when multiplied with a non-exact differential equation, makes it an exact differential equation.

Means: $(Non - Exact\ Equ.) \times I.F. = Exact\ Equ.$

**There can be more than one I.F. for one non-exact diff. equ.

Methods to find Integrating Factor (I.F.):

1. Inspection Method:

In this method we look for a particular expression: $(xdy - ydx)$ which is non-exact.

There are many I.F. available to make this expression an exact one, like;

$\frac{1}{x^2}, \frac{1}{y^2}, \frac{1}{xy}, \frac{1}{x^2+y^2}, \frac{1}{x^2-y^2}.$

We know: $(Non - Exact\ Equ.) \times I.F. = Exact\ Equ.$

1. $(xdy - ydx) \times \frac{1}{x^2} = \frac{(xdy-ydx)}{x^2} = d\left(\frac{y}{x}\right)$ (Quotient rule)
2. $(xdy - ydx) \times \frac{1}{y^2} = \frac{-(ydx-xdy)}{y^2} = -d\left(\frac{x}{y}\right)$ (Quotient rule)
3. $(xdy - ydx) \times \frac{1}{xy} = \frac{(xdy-ydx)}{xy} = d\left[\log\left(\frac{y}{x}\right)\right]$
4. $(xdy - ydx) \times \frac{1}{x^2+y^2} = \frac{(xdy-ydx)}{x^2+y^2} = d\left[tan^{-1}\left(\frac{y}{x}\right)\right]$
5. $(xdy - ydx) \times \frac{1}{x^2-y^2} = \frac{(xdy-ydx)}{x^2-y^2} = d\left[\frac{1}{2}\log\left(\frac{x+y}{x-y}\right)\right]$

Methods to find Integrating Factor (I.F.):

2. If $Mdx + Ndy = 0$ is a homogeneous differential equation in x and y, then:

I.F. = $\frac{1}{Mx+Ny}$

Look at equation: $(x^3 + y^3)dx - xy^2dy = 0$ (1)

Which is homogeneous in x and y

**An equation is said to be homogeneous is each term od M and N is of same degree.

Here $M = (x^3 + y^3)$. Each of these two terms x and y have same degree 3.

And $N = -x^1y^2$ is also of same degree 3 as sum of powers of x and y is 3

Thus, since equation (1) is homogeneous of degree 3.

So, I.F. = $\frac{1}{Mx+Ny}$

$\Rightarrow$I.F.= $\frac{1}{(x^3+y^3)x+(-xy^2)y} = \frac{1}{x^4}$

Methods to find I.F.

3. If an equation: $\mathrm{Mdx} + \mathrm{Ndy} = 0$

can be re-written as:

$$\mathrm{f(xy)ydx} + \mathrm{g(xy)xdy} = 0$$

then, $\mathrm{I.F.} = \frac{1}{\mathrm{Mx} - \mathrm{Ny}}$ provided $\mathrm{Mx} - \mathrm{Ny} \neq 0$

Methods to find I.F.

4. For an equation: $Mdx + Ndy = 0$ (1)

(I) If $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x)$, then $I.F. = e^{\int f(x)dx}$

(II) If $\dfrac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = f(y)$, then $I.F. = e^{\int f(y)dy}$

Clairaut's Equation:

An equation of the form: $y = px \pm f(p)$ (1)

is called Clairaut's equation where $p = \frac{dy}{dx}$

General solution of equation (1) is given by:

$y = cx \pm f(c)$ (Just replace p by c in the question, we get the answer)

* It is also called as a special case of equations solvable for y: $y = f(x, p)$

** It is a very good concept for asking MCQ

<u>Equations of First order and Higher Degree:</u>

1. <u>Equations solvable for p</u>: $p = f(x, y)$

We first replace $\frac{dy}{dx}$ $by\, p$ and then factorize the quadratic and cubic expressions.

Then we put each factor equal to zero and solve it. The separate solutions are combined to get general solution.

2. <u>Equations solvable for y</u>: y$= f(x, p)$

3. <u>Equations solvable for x</u>: x$= f(y, p)$

MCQ Practice Questions:

1. For what values of a, given differential equ.: $(x^2 - ay)dx + (y^2 - ax)dy = 0$ is exact:

(A) For all values of a

(B) There does not exist any value of a for which given diff. equ. is exact.

(C) Only for $a = 1$

(D) None of these.

Sol. Comparing with $Mdx + Ndy = 0$

$$M = (x^2 - ay) \Rightarrow \frac{\partial M}{\partial y} = -a, \qquad N = (y^2 - ax) \Rightarrow \frac{\partial N}{\partial x} = -a$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, So Given equation is exact for all values of a

Hence, Option (A) is right answer.

2. What is the relationship between a and b, so that the given equation:

$(x^8 + x + ay^2)dx + (y^8 - y + bxy)dy = 0$ is exact:

(A) $b = 2a$

(B) $a = b$

(C) $a \neq b$

(D) $a = 1, b = 3$

Sol. Comparing with $Mdx + Ndy = 0$

$$M = (x^8 + x + ay^2) \Rightarrow \frac{\partial M}{\partial y} = 2ay, \qquad N = (y^8 - y + bxy) \Rightarrow \frac{\partial N}{\partial x} = by$$

Since the given equation is exact, so $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ $\Rightarrow 2ay = by$ $\Rightarrow 2a = b$

Hence, Option (A) is right answer.

3. The differential equation: $M(x,y)dx + N(x,y)dy = 0$ will be an exact differential equation if:

(A) $\frac{\partial N}{\partial x} + \frac{\partial M}{\partial y} = 0$

(B) $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0$

(C) $\frac{\partial M}{\partial x} - \frac{\partial N}{\partial y} = 0$

(D) None of these

Sol. The necessary and sufficient condition for equation $Mdx + Ndy = 0$ to be an exact differential equation is: $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ $\Rightarrow \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0$

Hence, Option (B) is right answer.

4. In the non-exact differential equation: $M(x,y)dx + N(x,y)dy = 0$ if:

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x)$$ then Integrating factor is given by:

(A) $f(x)$

(B) $\log f(x)$

(C) $e^{f(x)}$

(D) $e^{\int f(x)dx}$

Sol. If $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x)$ then Integrating factor is given by: $I.F. = e^{\int f(x)dx}$

Hence, Option (D) is right answer.

5. Solution of differential equation: $xdy + ydx = 0$ is given by:

(A) $xy = c$

(B) $\frac{x}{y} = c$

(C) $x + y = c$

(D) $x - y = c$

Sol. The given equation is: $xdy + ydx = 0$

$\Rightarrow d(xy) = 0$ (Product rule of differentiation)

$\Rightarrow xy = c$ (Because derivative of constant is always zero)

Hence, Option (A) is right answer.

6. Which of the following is Clairaut's differential equation:

(A) $y = px + f(p)$

(B) $y = p^2x + f(p)$

(C) $y = p^2 + f(x)$

(D) None of these

Sol. Clairaut's equation is always of the form: $y = px \pm f(p)$

Hence, Option (A) is right answer.

7. Which of the following is I.F. for Clairaut's differential equation:

$(xdy - ydx) + a(x^2 + y^2)dx = 0$

(A) $\frac{1}{x^2}$

(B) $\frac{1}{y^2}$

(C) $\frac{1}{xy}$

(D) $\frac{1}{(x^2+y^2)}$

Sol. Here $\frac{1}{(x^2+y^2)}$ makes the equation exact: $\frac{(xdy-ydx)}{x^2+y^2} + adx = 0$

$\Rightarrow d\left[tan^{-1}\left(\frac{y}{x}\right)\right] + adx = 0 \qquad \Rightarrow tan^{-1}\left(\frac{y}{x}\right) + ax = c$ (On integrating)

Hence, Option (D) is right answer.

8. Which of the following is solution of differential equation: $xp^2 - yp + a = 0$

(A) $y = cx + \frac{a}{c}$

(B) $y = cx - e^c$

(C) $y = cx - sin^{-1}c$

(D) None of these

Sol. $xp^2 - yp + a = 0$

$\Rightarrow yp = xp^2 + a$

$\Rightarrow y = px + \frac{a}{p}$ Which is of Clairaut's form: $y = px + f(p)$

So, the general solution is given by putting $p = c$ i.e. $y = cx + \frac{a}{c}$

Hence, Option (A) is right answer.

9. Which of the following is sol. of equ.: $\sin px \cos y = \cos px \sin y + p$

(A) $y = cx + \frac{a}{c}$

(B) $y = cx - e^{c}$

(C) $y = cx - sin^{-1}c$

(D) None of these

Sol. $\Rightarrow \sin px \cos y - \cos px \sin y = p$

$\Rightarrow \sin(px - y) = p$

$\Rightarrow (px - y) = sin^{-1}p$

$\Rightarrow y = px - sin^{-1}p$

So, the general solution is given by putting $p = c$ i.e. $y = cx - sin^{-1}c$

Hence, Option (C) is right answer.

10. Which of the following is solution of: $(3x^2+6xy^2)dx+(6x^2y+4y^3)dy=0$

(A) $x^3-3xy+y^3=c$

(B) $x^3+3xy+y^4=c$

(C) $x^3-6x^2y-6xy^2+y^3=c$

(D) $x^3+3x^2y^2+y^4=c$

Sol. The given equation is exact as: $\frac{\partial M}{\partial y}=12xy=\frac{\partial N}{\partial x}$

Solution: $\int_{y=con.} M(x,y)dx+\int(Terms\ of\ N\ free\ from\ x)dy=C$

$\Rightarrow \int_{y=con.}(3x^2+6xy^2)dx+\int(4y^3)dy=C$

$\Rightarrow 3\frac{x^3}{3}+6\frac{x^2}{2}y^2+4\frac{y^4}{4}=C \qquad \Rightarrow x^3+3x^2y^2+y^4=c$

Hence, Option (D) is right answer.